Safety-Critical Control Under Timed Reach–Avoid Specifications: A Backup Control Barrier Function Approach

Yangrui Zhang, Tianjian Jiang, Zhongxuan Cai, and Yiding Ji, *Member, IEEE*

***Abstract*—Timed reach-avoid specifications, a subset of Signal Temporal Logic (STL), provide an expressive framework for specifying complex tasks in dynamic systems. This letter introduces a novel framework that encodes these specifications into a quadratic programming (QP) controller using backup control barrier functions (backup CBFs). Unlike existing approaches that rely on predefined safe sets, we utilize a backup control strategy within a prediction horizon to construct time-varying implicit sets online. Specifically, we address the limitation of standard backup CBFs regarding finite-time convergence by introducing contractive implicit sets. Rigorous analysis of the time-varying properties confirms that backup CBFs enforce system forward invariance within these contractive sets, driving real-time trajectories to satisfy given specifications. Our method unifies high-level planning with continuous control, while does not require explicit safe set computation for Eventually formulas or predefined performance functions. Numerical simulations validate the performance of our approach in safe planning and control of mobile robots.**

***Index Terms*—Control barrier functions, safety control, optimal control, formal methods, signal temporal logic.**

I. Introduction

With surging demands in real-world applications, robots are required to finish complicated tasks that accommodate both sequential and periodic requirements. This has led to a growing interest in developing control strategies that can effectively execute multi-objective tasks while ensuring safety and correctness [1]. Formal languages, such as linear temporal logic (LTL) and signal temporal logic (STL), provide an expressive framework for specifying intricate robotic tasks and behaviors [2], [3], [4].

Compared to LTL, STL is capable of formulating more complex tasks that incorporate continuous time constraints [5]. As a result, STL has been widely adopted to specify timed reach-avoid tasks for mobile robots [6], [7], [8]. Previous works on STL control typically encode specifications as mixed-integer constraints and synthesize controllers through mixed-integer programming [9], [10], [11]. However, these methods suffer from high computational complexity and poor scalability. Other approaches maximize STL robustness values in a receding horizon using model predictive control [12], [13], [14] or reinforcement learning [15], [16], often causing a separation between high-level planning and low-level implementation.

To alleviate these issues, control barrier functions (CBFs) are introduced to integrate STL specifications within an optimization framework [17], [18], [19], [20]. Reference [17] first proposes time-varying CBFs to encode STL specifications as hard constraints of quadratic programming. Reference [19] constructs the performance function in an online manner. These methods require a predicate function and an additional performance function for each atomic formula, causing control performance vulnerable to predefined temporal user behavior. Reference [20] utilizes control Lyapunov barrier functions (CLBFs) instead of CBFs, thus eliminating performance functions, but synthesizing CLBFs is more complex. References [21], [22] introduce a spatiotemporal tube-based approach that transforms STL into geometric constraints, facilitating systematic construction of smooth CBFs.

This work presents a novel framework that encodes timed reach-avoid specifications within an optimization problem via backup CBFs [23], [24], [25], [26], [27]. We employ an online backup control strategy to implicitly present the invariant set, thereby bypassing the computation of a predefined set. However, existing backup CBFs only ensure that the system reaches the backup set within a finite-time horizon following the backup control strategy, rather than under the real-time control signal. Reference [27] proposes a greedy policy-based control strategy to address this issue, but no convergence time is guaranteed. In contrast, we define a time-varying implicit set that gradually tightens over a finite-time period until it aligns with the backup set. Backup CBFs enforce the system's forward invariance within the implicit set, ensuring backup set finite-time reachable for real-time trajectories while

Received 8 November 2025; revised 17 January 2026; accepted 9 February 2026. Date of publication 17 February 2026; date of current version 5 March 2026. This work was supported in part by the National Natural Science Foundation of China under Grant 62303389 and Grant 62373289, in part by Guangdong Basic and Applied Basic Research Funding under Grant 2024A1515012586, in part by Guangdong Scientific Research Platform and Project Scheme under Grant 2024KTSCX039, and in part by the Youth Talent Support Program of Guangdong Association for Science and Technology under Grant SKXRC2025463. This article has not been presented at a conference. Recommended by Senior Editor A. P. Aguiar. *(Corresponding author: Yiding Ji.)*

Yangrui Zhang and Yiding Ji are with the Robotics and Autonomous Systems Thrust, Systems Hub, The Hong Kong University of Science and Technology (Guangzhou), Guangzhou 511453, China (e-mail: yzhang112@connect.hkust-gz.edu.cn; jiyiding@hkust-gz.edu.cn).

Tianjian Jiang and Zhongxuan Cai are with the College of Computer Science and Technology, National University of Defense Technology, Changsha 410073, China (e-mail: jiangtianjian@nudt.edu.cn; caizhongxuan@nudt.edu.cn).

Digital Object Identifier 10.1109/LCSYS.2026.3665795

adhering to the global safe set. Consequently, timed reach-avoid specifications are effectively encoded in time-varying implicit sets, without manipulation of temporal properties via predefined performance functions. Additionally, this approach unifies robotic planning and continuous control execution, providing a practical framework for encoding timed reach-avoid specifications for mobile robots.

This letter is structured as follows. Section II presents the necessary preliminaries and formulates the safety control problem under timed reach-avoid specifications. Section III discusses the forward invariance within time-varying and contractive sets that are implicitly defined through backup CBFs. Section IV incorporates the specifications into an optimal control problem, which is then addressed for controller synthesis. Numerical simulations in Section V validate the performance of our approach. Section VI draws the conclusion and states future research directions.

II. Preliminaries and Problem Formulation

In this letter, scalars are denoted by regular letters, and column vectors are represented by bold letters. $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}^n$ denote the set of real numbers, non-negative real numbers and real vectors of dimensions n, respectively.

Consider a control affine dynamic system modeled as:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}, \tag{1}$$

where $\boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathcal{U} \subseteq \mathbb{R}^m$ is a locally Lipschitz control input, and dynamic functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz functions.

A. Timed Reach-Avoid Specifications

The timed Reach-Avoid Specification is considered a subset of signal Temporal Logic, and is generally utilized in specifying reach-avoid task for mobile robots [6].

Signal temporal logic (STL) is a formal language that validates temporal properties consisting of predicates μ. For the signal $\boldsymbol{x}(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, μ is evaluated using a continuously differentiable predicate function $h_\mu(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\mu := \begin{cases} \text{True} & \text{if } h_\mu(\boldsymbol{x}) \geq 0, \\ \text{False} & \text{if } h_\mu(\boldsymbol{x}) < 0. \end{cases} \tag{2}$$

The syntax of STL defines an STL formula ϕ by

$$\phi ::= \text{True} \mid \mu \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \, U_{[a,b]} \, \phi_2, \tag{3}$$

where $\neg$ and $\wedge$ are the Boolean negation and conjunction operators. $\phi_1 \, U_{[a,b]} \, \phi_2$ is an Until operator that holds when ϕ_1 keeps true until ϕ_2 comes true. To restrict the analysis to timed reach-avoid specifications, additional operators are defined: Eventually $F_{[a,b]}\phi$:= True $U_{[a,b]}\phi$ and Always $G_{[a,b]}\phi$:= $\neg(F_{[a,b]}\neg\phi)$. The Eventually operator can be used to describe robots reaching target regions within specific intervals, while the Always operator can be adopted in scenarios where robots continuously avoid forbidden areas throughout the specified period. The corresponding semantics are introduced as follows.

Definition 1 (Semantics of Timed Reach-Avoid Specifications [6]): For a given signal $\boldsymbol{x}(t) \in \mathbb{R}^n$, the satisfaction relation of timed reach-avoid formulas is recursively defined by:

$$\begin{aligned}
(\boldsymbol{x}, t) &\models \mu && \Leftrightarrow h_\mu(\boldsymbol{x}) \geq 0 \\
(\boldsymbol{x}, t) &\models \neg\phi && \Leftrightarrow \neg((\boldsymbol{x}, t) \models \phi) \\
(\boldsymbol{x}, t) &\models \phi_1 \wedge \phi_2 && \Leftrightarrow (\boldsymbol{x}, t) \models \phi_1 \wedge (x, t) \models \phi_2 \\
(\boldsymbol{x}, t) &\models F_{[a,b]}\phi && \Leftrightarrow \exists t_1 \in [t + a, t + b] \text{s.t. } (\boldsymbol{x}, t_1) \models \phi \\
(\boldsymbol{x}, t) &\models G_{[a,b]}\phi && \Leftrightarrow \forall t_1 \in [t + a, t + b], (\boldsymbol{x}, t_1) \models \phi.
\end{aligned}$$

B. Time-Varying Control Barrier Functions

Time-varying control barrier functions (CBFs) are utilized to ensure that the system (1) evolves within a safe set $\mathcal{C}(t)$. Define the differentiable function $h : \mathcal{X} \times [t_0, t_1] \to \mathbb{R}$, and the safe set is defined by its superlevel set as $\mathcal{C}(t) := \{\boldsymbol{x} \in \mathcal{X} | h(\boldsymbol{x}, t) \geq 0\}$. We review the following concepts to introduce the safety-critical control method via CBFs.

Definition 2 (Forward Invariance): A set $\mathcal{C}(t)$ is forward invariant under control input $\boldsymbol{u}$ if for all initial state $\boldsymbol{x_0} \in \mathcal{C}(t_0)$, there exists a continuous trajectory $\boldsymbol{x} : [t_0, t_1] \to \mathbb{R}^n$ with $\boldsymbol{x}(t_0) := \boldsymbol{x_0}$ such that $\boldsymbol{x}(t) \in \mathcal{C}(t)$ for all $t \in [t_0, t_1]$.

Definition 3: [Class $\mathcal{K}$ functions] A continuous function α: $\mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is called a class $\mathcal{K}$ function if it is strictly increasing with $\alpha(0) = 0$.

Definition 4: [Time-varying Control Barrier Functions [17]] $h : \mathcal{X} \times [t_0, t_1] \to \mathbb{R}$ is a control barrier function (CBF) on the safety set $\mathcal{C}(t)$ if there exists a class $\mathcal{K}$ function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all $(\boldsymbol{x}, t) \in \mathcal{C}(t)$, the condition below holds:

$$\sup_{\boldsymbol{u}\in\mathcal{U}} \left[\frac{\partial h(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}^\top (f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}) + \frac{\partial h(\boldsymbol{x}, t)}{\partial t} \right] \geq -\alpha(h(\boldsymbol{x}, t))$$

The results of [17] indicate that if $h(\boldsymbol{x}, t)$ is a CBF, then it can guarantee forward invariance for given systems.

Lemma 1 (Theorem 1. in [17]): Assume that the control input $\boldsymbol{u} \in \mathcal{U}$ is locally Lipschitz continuous in $\boldsymbol{x}$ and piecewise continuous in t. The unique solutions to (1) are defined in $[t_0, t_1]$. If there exists a CBF $h(\boldsymbol{x}, t)$ that satisfies Definition 4 for $\boldsymbol{u}$, then the set $\mathcal{C}(t)$ is forward invariant under $\boldsymbol{u}$.

In this letter, CBFs are integrated into an online filter based on quadratic programming (QP). The QP filter aims to minimize the difference between the desired inputs $\boldsymbol{u}_{ref}$ from high-level commands and the actual inputs $\boldsymbol{u}$. This approach optimizes the fidelity of $\boldsymbol{u}_{ref}$ while ensuring system safety.

C. Online Forward Invariance via Backup CBFs

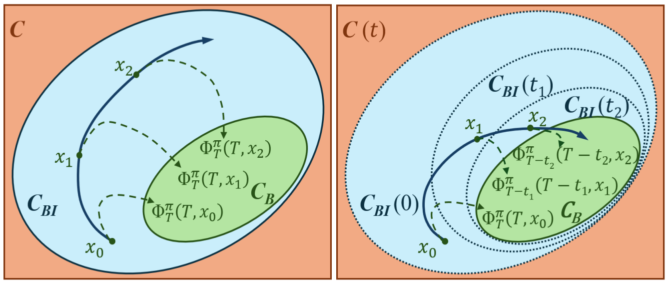
Backup CBFs are introduced to maintain the system (1) within an arbitrary set constructed online, rather than relying on a predefined safe set [23], [26]. Let $\mathcal{C} := \{\boldsymbol{x} \in \mathcal{X} | h(\boldsymbol{x}) \geq 0\}$ be a safe set in which the system (1) should remain forward invariant, and $\mathcal{C}_B := \{\boldsymbol{x} \in \mathcal{X} | h_b(\boldsymbol{x}) \geq 0\}$ be a backup set that $\mathcal{C}_B \subseteq \mathcal{C}$. The set $\mathcal{C}_{BI}$ is implicitly defined through a backup control strategy. Although the backup strategy is never directly implemented, it is utilized to synthesize safety certificates. For a locally Lipschitz backup control law $\boldsymbol{\pi} : \mathcal{X} \to \mathcal{U}$, the closed-loop system (1) under $\boldsymbol{\pi}(\boldsymbol{x})$ is described as

$$f_{cl}(\boldsymbol{x}) \triangleq f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{\pi}(\boldsymbol{x}). \tag{4}$$

Then the flow map of (1) over the time interval $[0, T]$ under $\boldsymbol{\pi}(\boldsymbol{x})$ is represented as

$$\dot{\Phi}^{\pi}_T(\tau, \boldsymbol{x}) = f_{cl}(\Phi^{\pi}_T(\tau, \boldsymbol{x})), \Phi^{\pi}_T(0, \boldsymbol{x}) = \boldsymbol{x}, \tag{5}$$

where $\tau \in [0, T]$. $\Phi^{\pi}_T(\tau, \boldsymbol{x})$ is the solution of (4) at τ, and $\partial\Phi^{\pi}_T(\tau, \boldsymbol{x})/\partial\boldsymbol{x}$ indicates the sensitivity of the flow (5) in the initial condition $\boldsymbol{x}$. $\Phi^{\pi}_T(\tau, \boldsymbol{x})$ is additive in t that satisfies



(a) (b)

Fig. 1. (a) $\mathcal{C}_{BI}$ is an implicit set that enlarges the known set $\mathcal{C}_B$ with backup policy, $\boldsymbol{x} \in \mathcal{C}_{BI}$ maintains forward invariant in $\mathcal{C}_B$ and T-time reachable for $\mathcal{C}_B$. (b) $\mathcal{C}_{BI}(t)$ is a contractive set over time T. If $\boldsymbol{x}$ maintains forward invariant in $\mathcal{C}_{BI}(t)$, it would be ensured to reach $\mathcal{C}_B$ when $\mathcal{C}_{BI}(t)$ is tightened and aligns with $\mathcal{C}_B$ at time T.

$\Phi_T^\pi(\Phi_T^\pi(x,\tau_1),\tau_2) = \Phi_T^\pi(x,\tau_1+\tau_2)$. The online forward invariance set $\mathcal{C}_{BI}$, which satisfies $\mathcal{C}_B \subseteq \mathcal{C}_{BI} \subseteq \mathcal{C}$, is implicitly defined from the flow map as

$$\mathcal{C}_{BI} \triangleq \{\boldsymbol{x} \in \mathcal{X} | \Phi_T^{\boldsymbol{\pi}}(T,\boldsymbol{x}) \in \mathcal{C}_B \wedge \forall \tau \in [0,T], \Phi_T^{\boldsymbol{\pi}}(\tau,\boldsymbol{x}) \in \mathcal{C}\}. \quad (6)$$

Assuming that the backup control policy $\boldsymbol{\pi}(\boldsymbol{x})$ ensures the forward invariance of $\mathcal{C}_{BI}$, i.e., $\boldsymbol{\pi}(\boldsymbol{x})$ guides (1) towards $\mathcal{C}_B$ within T-time horizon while maintaining forward invariance in $\mathcal{C}$, the main result of the backup CBFs is presented.

Lemma 2: [*Proposition 2. in* [24]] Given the system (1), if there exists a class $\mathcal{K}$ function α such that for all $\boldsymbol{x} \in \mathcal{C}$ and $\tau \in [0,T]$, the following condition holds:

$$\frac{dh_b(\Phi_T^{\boldsymbol{\pi}}(T,\boldsymbol{x}))}{dt}(\boldsymbol{x},\boldsymbol{u}) \geq -\alpha(h_b(\Phi_T^{\boldsymbol{\pi}}(T,\boldsymbol{x}))), \quad (7a)$$

$$\frac{dh(\Phi_T^{\boldsymbol{\pi}}(\tau,\boldsymbol{x}))}{dt}(\boldsymbol{x},\boldsymbol{u}) \geq -\alpha(h(\Phi_T^{\boldsymbol{\pi}}(\tau,\boldsymbol{x}))), \quad (7b)$$

then the forward invariance of $\mathcal{C}_{BI}$ is guaranteed.

For static backup sets and safe sets in existing work [23], [24], [25], the backup CBF constraints are formulated as follows:

$$\frac{dh_b}{d\boldsymbol{x}}\frac{\partial\Phi_T^{\boldsymbol{\pi}}(T,\boldsymbol{x})}{\partial\boldsymbol{x}}\dot{\boldsymbol{x}} \geq -\alpha(h_b(\Phi_T^{\boldsymbol{\pi}}(T,\boldsymbol{x}))), \quad (8a)$$

$$\frac{dh}{d\boldsymbol{x}}\frac{\partial\Phi_T^{\boldsymbol{\pi}}(\tau,\boldsymbol{x})}{\partial\boldsymbol{x}}\dot{\boldsymbol{x}} \geq -\alpha(h(\Phi_T^{\boldsymbol{\pi}}(\tau,\boldsymbol{x}))), \quad (8b)$$

where $\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}$, and $\partial\Phi_T^{\boldsymbol{\pi}}(T,\boldsymbol{x})/\partial\boldsymbol{x}$ is the sensitivity Jacobian at $\boldsymbol{x}$. These constraints ensure that the system (1) remains in the safe set $\mathcal{C}$, and maintain the capability to reach $\mathcal{C}_B$ within a finite time T, which is shown in Fig. 1a pictorially.

D. Problem Formulation

This letter focuses on timed reach-avoid specifications for the control of mobile robots. The problem is formulated below.

Problem 1 (Safety-Critical Controller Synthesis): Given the system in (1), and the timed reach-avoid specification below:

$$\psi ::= \text{True} \mid \mu \mid \neg\mu \mid \psi_1 \wedge \psi_2, \quad (9a)$$

$$\phi ::= F_{[a,b]}\psi \mid G_{[a,b]}\psi \mid \phi_1 \wedge \phi_2. \quad (9b)$$

Denote by $\widetilde{\mathcal{C}}$ the safe set for (1) whose trajectories satisfy the above specification, then a safety filter should be given such that the trajectory $\boldsymbol{x}$ satisfies ϕ for all time $t \geq 0$:

$$\min \parallel \boldsymbol{u}_{ref} - \boldsymbol{u} \parallel^2 \quad (10a)$$

$$\text{s.t.} \quad \boldsymbol{x} \in \widetilde{\mathcal{C}} \in \mathcal{X}, \boldsymbol{u} \in \mathcal{U}. \quad (10b)$$

III. Online Forward Invariance Theory of Contractive Sets

This section introduces the online forward invariance theory of contractive implicit sets. This result ensures that real-time trajectories safely reach a specified set within a fixed time frame, which is a complement to forward invariance in [23].

The backup CBFs in [23] enforce the system remain in an implicit set $\mathcal{C}_{BI}$. That is, for all $\boldsymbol{x} \in \mathcal{X}$, the backup trajectories reach $\mathcal{C}_B$ within a fixed time T under a certain backup controller $\boldsymbol{\pi}(\boldsymbol{x})$ while always maintaining in the safe set $\mathcal{C}$. However, it cannot drive the real-time trajectory to $\mathcal{C}_B$, nor does it guarantee finite-time reachability. To address this issue, we propose that if the implicit set $\mathcal{C}_{BI}(t)$ is a time-varying set that contracts by decreasing the backup horizon until $\mathcal{C}_{BI}(t)$ aligns with $\mathcal{C}_B$, and the system remains forward invariant in $\mathcal{C}_{BI}(t)$, then its trajectories reach $\mathcal{C}_B$ in finite time.

Proposition 1: Given system (1), a fixed backup set $\mathcal{C}_B$ and a time-varying safe set $\mathcal{C}(t) := \{\boldsymbol{x} \in \mathcal{X} | h(\boldsymbol{x},t) \geq 0\}$, where $h(\boldsymbol{x},t)$ is absolutely continuous in $\boldsymbol{x}$. For all $\boldsymbol{x}_0 \in \mathcal{C}(0)$, if there exists control input $\boldsymbol{u}$ at time $t \in [0,T]$ such that (1) remains forward invariant in a time-varying arbitrary set

$$\mathcal{C}_{BI}(t) \triangleq \{\boldsymbol{x} \in \mathcal{X} | \Phi_{T-t}^{\boldsymbol{\pi}}(T-t,\boldsymbol{x}) \in \mathcal{C}_B \wedge \forall \tau \in [0,T-t], \Phi_{T-t}^{\boldsymbol{\pi}}(\tau,\boldsymbol{x}) \in \mathcal{C}(t)\}, \quad (11)$$

then $\mathcal{C}_B$ is T-time reachable for (1).

Proof: The time-varying set (11) at time T is $\mathcal{C}_{BI}(T) \triangleq \{\boldsymbol{x} \in \mathcal{X} | \Phi_0^{\boldsymbol{\pi}}(0,\boldsymbol{x}) \in \mathcal{C}_B \wedge \Phi_0^{\boldsymbol{\pi}}(0,\boldsymbol{x}) \in \mathcal{C}(t)\}$. From (5), $\Phi_{T-t}^{\boldsymbol{\pi}}(0,\boldsymbol{x}) = \boldsymbol{x}$ for all $t \in [0,T]$, thus $\boldsymbol{x} \in \mathcal{C}_{BI}(T) \Leftrightarrow \boldsymbol{x} \in \mathcal{C}_B \wedge \boldsymbol{x} \in \mathcal{C}(t)$. Since (1) remains forward invariant within $\mathcal{C}_{BI}(t)$ over $[0,T]$, it is guaranteed to reach $\mathcal{C}_B$ at time T. ■

Fig. 1b illustrates Proposition 1. Our goal is to design a control strategy $\boldsymbol{u}$ that keeps the system (1) within the contractive set. Since the contractive set defined by (11) imposes a time-varying constraint on the system, previous online reachability theory is insufficient for the design.

Proposition 2: For $\boldsymbol{x}_{t'} \xrightarrow{\boldsymbol{u}_{t'}} \boldsymbol{x}_{t''}$ where $t', t'' \in [0,T]$, (8a) is insufficient to ensure (1) forward invariant within $\mathcal{C}_{BI}(t)$.

Proof: Since $\mathcal{C}_{BI}(t)$ is defined in (11) with a backup horizon of $T-t$, for any $t', t'' \in [0,T]$ such that $t'' > t'$, it follows that $T - t'' < T - t'$, which implies that the constraints defining $\mathcal{C}_{BI}(t'')$ are imposed over a strictly shorter backup horizon than those defining $\mathcal{C}_{BI}(t')$. Consequently, any $\boldsymbol{x}$ whose backup trajectory satisfies the terminal and safety conditions over $[0, T-t'']$ also satisfies the corresponding conditions over the prefix interval $[0, T-t']$. Also, the implicit sets are contractive and satisfy $\mathcal{C}_{BI}(t'') \subset \mathcal{C}_{BI}(t')$. Following *Lemma 2*, a given $\boldsymbol{u}_{t'}$ that satisfies (8a) only renders $\boldsymbol{x}_{t''} \in \mathcal{C}_{BI}(t')$ instead of $x_{t''} \in C_{BI}(t'')$, and it possibly drives $\boldsymbol{x}_{t''}$ into $\mathcal{C}_{BI}(t'')\backslash\mathcal{C}_{BI}(t')$, which violates forward invariance of $\mathcal{C}_{BI}(t)$. ■

Proposition 2 indicates that for all $\boldsymbol{x} \in \mathcal{X}$ at time t, if (8a) is implemented, then the backup controller with hard constraints $\Phi_{T-t}^{\boldsymbol{\pi}}(T-t,\boldsymbol{x}) \in \mathcal{C}_B$ may encounter infeasibility due to the strict requirement to reach the backup set $\mathcal{C}_B$. Therefore, the backup CBF-based filter in the previous works [23], [24] fail to ensure online forward invariance for contractive sets of (11).

To maintain the system within $\mathcal{C}_{BI}(t)$ and reach $\mathcal{C}_B$ in a finite horizon T, we consider time-varying properties of backup CBFs. Reference [25] marks that implicit safe sets can be time-varying even if the initial sets are static. In [27], predefined time-varying sets may render the backup controller infeasible and a modified backup CBF strategy is proposed to address this issue. The time-varying condition in Proposition 1 differs from them since the prior backup set $\mathcal{C}_B$ is fixed, whereas the predictive horizon of the backup set varies. As a result, a new framework is required to synthesize backup CBFs for (11).

By (5), the flow map over a contractive horizon is given as

$$\Phi^{\pi}_{T-t}(\tau, \boldsymbol{x}) = \boldsymbol{x} + \int_{\hat{\tau}=0}^{\tau} f_{cl}(\Phi^{\pi}_{T-t}(\hat{\tau}, \boldsymbol{x}))d\hat{\tau},$$

and its derivative of t is given by the chain rule with

$$\frac{d\Phi^{\pi}_{T-t}(\tau, \boldsymbol{x})}{dt} = \frac{\partial\Phi^{\pi}_{T-t}(\tau, \boldsymbol{x})}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} + \frac{\partial\Phi^{\pi}_{T-t}(\tau, \boldsymbol{x})}{\partial t}.$$

When time-varying properties of $\Phi^{\pi}_{T-t}(T-t, \boldsymbol{x}))$ are considered, the left hand in inequality (7a) becomes

$$\begin{aligned} &\frac{dh_b(\Phi^{\pi}_{T-t}(T-t, \boldsymbol{x}))}{dt}(\boldsymbol{x}, \boldsymbol{u}) \\ &= \frac{dh_b}{d\boldsymbol{x}}\left(\frac{\partial\Phi^{\pi}_{T-t}(T-t, \boldsymbol{x})}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} + \frac{\partial\Phi^{\pi}_{T-t}(T-t, \boldsymbol{x})}{\partial(T-t)}\frac{\partial(T-t)}{\partial t}\right) \\ &= \frac{dh_b}{d\boldsymbol{x}}\left(\frac{\partial\Phi^{\pi}_{T-t}(T-t, \boldsymbol{x})}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} - f_{cl}(\Phi^{\pi}_{T-t}(T-t, \boldsymbol{x}))\right). \end{aligned} \quad (12)$$

In (12), $-f_{cl}(\Phi^{\pi}_{T-t}(T-t, \boldsymbol{x}))$ represents the time-varying property of the backup strategy caused by a contractive horizon, which is influenced by $\boldsymbol{\pi}(\boldsymbol{x})$. Next, consider a time-varying safe set $\mathcal{C}(t)$, the left hand in inequality (7b) becomes

$$\begin{aligned} &\frac{dh(\Phi^{\pi}_{T-t}(\tau, \boldsymbol{x}))}{dt}(\boldsymbol{x}, \boldsymbol{u}) \\ &= \frac{dh}{d\boldsymbol{x}}\frac{\partial\Phi^{\pi}_{T-t}(\tau, \boldsymbol{x})}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} + \frac{\partial h(\Phi^{\pi}_{T-t}(\tau, \boldsymbol{x}))}{\partial t}. \end{aligned} \quad (13)$$

Remark 1: For a non-contractive set $\mathcal{C}_{BI}$, a fixed backup horizon T turns $\partial\Phi^{\pi}_{T}(T, \boldsymbol{x})/\partial t = 0$ in (12), then the reachability constraint (12) reduces to (8a). Meanwhile, (13) shares the same expression as the back up CBFs constraint for time-varying safe sets $\mathcal{C}_B(t)$ in [27].

Then, we present the main result for the online forward invariance theory of contractive sets.

Theorem 1: Given system (1) and a contractive set (11), if there exist a class $\mathcal{K}$ function α and control input $\boldsymbol{u}(t)$ such that $\forall \boldsymbol{x}_0 \in \mathcal{C}(t)$, $t \in [0, T]$, $\tau \in [0, T-t]$, the condition holds:

$$\begin{aligned} &\frac{dh_b}{d\boldsymbol{x}}\left(\frac{\partial\Phi^{\pi}_{T-t}(T-t, \boldsymbol{x})}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} - f_{cl}(\Phi^{\pi}_{T-t}(T-t, \boldsymbol{x}))\right. \\ &\geq -\alpha(h_b(\Phi^{\pi}_{T-t}(T-t, \boldsymbol{x}))), \end{aligned} \quad (14a)$$

$$\begin{aligned} &\frac{dh}{d\boldsymbol{x}}\frac{\partial\Phi^{\pi}_{T-t}(\tau, \boldsymbol{x})}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} + \frac{\partial h(\Phi^{\pi}_{T-t}(\tau, \boldsymbol{x}))}{\partial t} \\ &\geq -\alpha(h(\Phi^{\pi}_{T-t}(\tau, \boldsymbol{x}))), \end{aligned} \quad (14b)$$

then the forward invariance of the contractive set $\mathcal{C}_{BI}(t)$ is guaranteed, and $\boldsymbol{x}$ reaches $\mathcal{C}_B$ within T-time horizon.

Proof: From (12), (13) and Lemma 2, $\mathcal{C}_{BI}(t)$ is forward invariant under control input $\boldsymbol{u}$. Then according to Proposition 1, $\mathcal{C}_{BI}(t)$ is T-time reachable. ■

IV. Backup Control Barrier Functions for Timed Reach-Avoid Specifications

In this section, we design backup CBFs to accomplish timed reach-avoid tasks. These tasks are first decomposed into Always and Eventually formulas, respectively. Composed Always formulas are satisfied when the system maintains within a time-varying safe set $\mathcal{C}_G(t) := \{\boldsymbol{x} \in \mathcal{X} | h_G(\boldsymbol{x}, t) \geq 0\}$. An Eventually formula holds when a backup set $\mathcal{C}_{F_i}(t) := \{\boldsymbol{x} \in \mathcal{X} | h_{F_i}(\boldsymbol{x}, t) \geq 0\}$ is reached within a finite horizon.

A. Always Formulas

For each Always formula $\phi_{G_j} = G_{[a_j, b_j]}\psi_j$, the trajectory of $\boldsymbol{x}$ should satisfy $\forall t \in [a_j, b_j], h_{\psi_j}(\boldsymbol{x}) \geq 0$. Inspired by [17], we introduce time varying CBFs with hybrid switching.

Assumption 1: $\forall \phi_{G_j} = G_{[a_j, b_j]}\psi_j$, $h_{\psi_j}(\boldsymbol{x}(a_j)) \geq 0$ holds.

Theorem 2: Given a unit step function $\theta(t)$, a candidate CBF

$$h_{G_j}(\boldsymbol{x}, t) = [\theta(t-a_j) - \theta(t-b_j)]h_{\psi_j}(\boldsymbol{x}) \quad (15)$$

is continuous in $\boldsymbol{x}$ and piecewise continuous in t, with the safe set $\mathcal{C}_{G_j}(t) := \{\boldsymbol{x} \in \mathcal{X} | h_{G_j}(\boldsymbol{x}, t) \geq 0\}$. Then for all $\boldsymbol{x}(0) \in \mathcal{C}_{G_j}(0)$, $h_{G_j}(\boldsymbol{x}, t)$ is a valid CBF enforcing $(\boldsymbol{x}, t) \models G_{[a_j, b_j]}\psi_j$ under Assumption 1.

Proof: By construction, $h_{G_j}(\boldsymbol{x}, t)$ is piecewise with switching instants at a_j and b_j. For $t \in (a, b)$, $h_{G_j}(\boldsymbol{x}, t)$ is a time-invariant CBF that maintains (1) forward invariance within $\mathcal{C}_{\psi_j} := \{\boldsymbol{x} \in \mathcal{X} | h_{\psi_j}(\boldsymbol{x}) \geq 0\}$. For $t \in [0, a_j) \cup (b_j, +\infty)$, $h_{G_j}(\boldsymbol{x}, t) = dh_{G_j}/dx = dh_{G_j}/dt = 0$ always holds, thus inequality $\dot{h}_{G_j} \geq -\alpha(h_{G_j})$ holds on these intervals. For $t = a_j$, $\mathcal{C}_{G_j}(t)$ switches from $\mathcal{X}$ to $\mathcal{C}_{\psi_j}$. By Assumption 1, $h_{\psi_j}(\boldsymbol{x}(a_j)) \geq 0$ implies that $\boldsymbol{x}(a_j) \in \mathcal{C}^{+}_{G_j}(a_j)$. Hence, the system state is consistent with the post-switch safe set, and no violation of forward invariance occurs at the switching instant. For $t = b_j$, $\mathcal{C}_{G_j}(t)$ switches from $\mathcal{C}_{\psi_j}$ to $\mathcal{X}$. Since $\mathcal{C}_{G_j}(t)$ is non-decreasing with $\mathcal{C}^{-}_{G_j}(b_j) \subseteq \mathcal{C}^{+}_{G_j}(b_j)$, i.e., any $\boldsymbol{x}^{-}(b_j) \in \mathcal{C}^{-}_{G_j}(b_j)$ also satisfies $\boldsymbol{x}^{+}(b_j) \in \mathcal{C}^{+}_{G_j}(b_j)$, then forward invariance is trivially preserved across this switching instant according to Theorem 2 in [17] (The superscript $+, -$ indicates the values before and after the switch, respectively). ■

Let $\phi_G := \bigwedge_{j=1}^{n} \phi_{G_j}$ denote the conjunction of n Always formulas, which is utilized to synthesize the largest safe set $\mathcal{C}_G(t) := \{\boldsymbol{x} \in \mathcal{X} | h_G(\boldsymbol{x}, t) \geq 0\}$ for system (1). By Theorem 2, we obtain CBFs $h_{G_j}(\boldsymbol{x}, t)$ for individual Always formula ϕ_{G_j}. The CBF for $\mathcal{C}_G(t)$ is then computed via a smooth under-approximation of the conjunction operator [17]:

$$h_G(\boldsymbol{x}, t) = -\frac{1}{\kappa}\ln\sum_{j=1}^{n} e^{-\kappa h_{G_j}(\boldsymbol{x}, t)}, \quad (16)$$

where $\kappa > 0$ is a smoothening parameter.

B. Eventually Formulas

For each Eventually formula $\phi_{F_i} = F_{[a_i, b_i]}\psi_i$, the trajectory $\boldsymbol{x}$ should satisfy $\exists t \in [a_i, b_i]$ such that $h_{\psi_i}(\boldsymbol{x}) \geq 0$. Our goal is to synthesize a control strategy $\boldsymbol{u}$ so that $\boldsymbol{x}$ reaches $C_{\psi_i} := h_{\psi_i}(\boldsymbol{x}) \geq 0$ within a finite-time horizon to achieve $F_{[a_i, b_i]}\psi_i$. The maximum converge time T satisfies $T \geq b_i - a_i$.

Initially, we consider $C_{\psi_i} := \{\boldsymbol{x} \in \mathcal{X} | h_{\psi_i}(\boldsymbol{x}) \geq 0\}$ as the backup set. As highlighted in [23], the backup controller maximizes reachable sets. Therefore, we adopt the model predictive controller (MPC) in [12] as the backup controller to maximize the cumulative robustness measure of given formulas. Our controller is defined over the interval $[t_{0i}, b_i]$, where $t_{0i} < a_i < b_i$. With a time-varying backup horizon $N_i(t) = \{T, t \in [t_{0i}, b_i - T]; b_i - t, t \in (b_i - T, b_i]\}$, the implicitly reachable set becomes contractive. $F_{[a_i, b_i]}\psi_i$ is enforced by imposing $h_{\psi_i}(\boldsymbol{x}) \geq 0$ as hard constraints at the end of the backup horizon. Then the backup controller is:

MPC–Backup Controller

$$J^* = \max_{\boldsymbol{\pi}_{0:N_i-1}} \beta_{N_i} h_{\psi_i}(\boldsymbol{x}_{N_i}) + \sum_{\tau_i=0}^{N_i-1} \beta_{\tau_i} h_{\psi_i}(\boldsymbol{x}_{\tau_i}) \tag{17a}$$

$$\text{s.t.} \quad \boldsymbol{x}_{\tau_i+1} = \boldsymbol{x}_{\tau_i} + f_d(\boldsymbol{x}_{\tau_i}, \boldsymbol{\pi}_{\tau_i}), \tag{17b}$$

$$\boldsymbol{x}_{\tau_i} \in \mathcal{X}, \boldsymbol{\pi}_{\tau_i} \in \mathcal{U}, \tag{17c}$$

$$\boldsymbol{x}_0 = \boldsymbol{x}(t), \tag{17d}$$

$$h_{\psi_i}(\boldsymbol{x}_{N_i}) \geq 0 \Rightarrow F_{[a_i,b_i]}\psi_i \models \text{True}. \tag{17e}$$

where $\tau_i = 0, 1, \ldots, N_i$. (17b) describes the dynamic of the discretized system from (1); for parameters $\beta_{0:N_i}$ in (17a), β_{N_i} is much larger than the other β_{τ_i}.

The backup control strategy generates the flow that satisfies $\Phi^{\boldsymbol{\pi}}_{N_i}(N_i, \boldsymbol{x}) \in \mathcal{C}_{\psi_i}$. So we formulate CBF candidate $h_{F_i}(\boldsymbol{x}) := h_{\psi_i}(\boldsymbol{x})$. According to Theorem 1, the satisfaction of $F_{[a_i,b_i]}\psi_i$ for (1) is guaranteed by the following inequality:

$$\frac{dh_{F_i}}{dx}\left(\frac{\partial \Phi^{\boldsymbol{\pi}}_{N_i}(N_i, \boldsymbol{x})}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} - \epsilon_i f_{cl}(\Phi^{\boldsymbol{\pi}}_{N_i}(N_i, \boldsymbol{x}), \boldsymbol{x})\right) \geq -\alpha(h_{F_i}(\Phi^{\boldsymbol{\pi}}_{N_i}(N_i, \boldsymbol{x}))), \tag{18}$$

where $\epsilon_i = \{0, t \in [t_{0i}, b_i - T]; 1, t \in (b_i - T, b_i]\}$ is a parameter that denotes whether the set $\mathcal{C}_{F_iI}(t) = \{\boldsymbol{x} \in \mathcal{X} | \Phi^{\boldsymbol{\pi}}_{N_i}(N_i, \boldsymbol{x}), \boldsymbol{x}) \in \mathcal{C}_{\psi_i}\}$ starts to contract.

Remark 2: As $\mathcal{C}_{F_iI}(t)$ for each ϕ_{F_i} are obtained by independent backup controllers with distinct horizons and flow functions, it is challenging to compose the implicit sets like the explicit ones in (16). As a result, we use backup CBFs to represent each ϕ_{F_i} individually instead of a composed ϕ_F.

C. Controller Synthesis

We synthesize a QP controller as a safety filter to solve Problem 1. Consider a formula $\phi = \phi_G \wedge \phi_F$ where $\phi_G = \bigwedge_{i=1}^{n} \phi_{G_j}$ and $\phi_F = \bigwedge_{i=1}^{m} \phi_{F_i}$, system (1) should ensure finite-time reachability for each $\mathcal{C}_{F_i}$ and maintain forward invariance in the global safe set $\mathcal{C}_G$. The backup control strategies satisfying the above requirements are presented as:

MPC–Backup Controller with ϕ_G constraints

$$J^* = \max_{\boldsymbol{\pi}_{0_i:N_i-1}} \beta_{N_i} h_{\psi_i}(\boldsymbol{x}_{N_i}) + \sum_{\tau_i=0}^{N_i-1} \beta_{\tau_i} h_{\psi_i}(\boldsymbol{x}_{\tau_i}) \tag{19a}$$

$$\text{s.t.} \quad (16b), (16c), (16d), (16e),$$

$$\frac{\partial h_G}{\partial \boldsymbol{x}}^{\top} f_{cl}(\boldsymbol{x}_{\tau_i}, \boldsymbol{\pi}_{\tau_i}) + \frac{\partial h_G}{\partial t} \geq -\alpha(h_G). \tag{19b}$$

Remark 3: Though $\Phi^{\boldsymbol{\pi}}_{N_i}(N_i, \boldsymbol{x})$ generated by MPC is only piecewise continuous in $\boldsymbol{x}$ and t, the approximation error is bound by $|h_{F_i}(\Phi^{\boldsymbol{\pi}_i}_{N_i}(\tau, \boldsymbol{x})) - h_{F_i}(\Phi^{\boldsymbol{\pi}_i}_{N_i}(\tau_i, \boldsymbol{x}))| \leq L_{F_i}|\tau - \tau_i|$, $|h_G(\Phi^{\boldsymbol{\pi}_i}_{N_i}(\tau, \boldsymbol{x})) - h_G(\Phi^{\boldsymbol{\pi}_i}_{N_i}(\tau_i, \boldsymbol{x}))| \leq L_G|\tau - \tau_i|$ from [24].

To ensure satisfaction of ϕ, the system (1) should maintain forward invariance within the implicit set $\mathcal{C}(t)$ defined as:

$$\mathcal{C}(t) := \bigcap_{i=1}^{m} \mathcal{C}_i(t), \tag{20a}$$

$$\mathcal{C}_i(t) := \{\boldsymbol{x} \in \mathcal{X} | \Phi^{\boldsymbol{\pi}_i}_{N_i}(N_i, \boldsymbol{x}) \in \mathcal{C}_{F_i} \wedge \forall \tau_i \in [0, N_i], \Phi^{\boldsymbol{\pi}_i}_{N_i}(\tau_i, \boldsymbol{x}) \in \mathcal{C}_G\}, \tag{20b}$$

where (1) being forward invariance in $\mathcal{C}_i(t)$ ensures satisfaction of ϕ_{F_i} and following ϕ_G. Given a positive semi-definite matrix $Q \in \mathbb{R}^{m \times m}$, the online QP filter for ϕ is designed as:

QP Filter

$$\min (\boldsymbol{u}_{ref} - \boldsymbol{u})^{\top} Q(\boldsymbol{u}_{ref} - \boldsymbol{u})$$

$$\text{s.t.} \quad \forall i \in [1, m],$$

$$\frac{dh_{F_i}}{dx}\left(\frac{\partial \Phi^{\boldsymbol{\pi}_i}_{N_i}(N_i, \boldsymbol{x})}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} - \epsilon_i f_{cl}(\Phi^{\boldsymbol{\pi}_i}_{N_i}(N_i, \boldsymbol{x}), \boldsymbol{x})\right) \geq -\alpha(h_{F_i}(\Phi^{\boldsymbol{\pi}_i}_{N_i}(N_i, \boldsymbol{x}))), \tag{21a}$$

$$\forall \tau_i \in [0, N_i],$$

$$\frac{dh_G}{dx}\frac{\partial \Phi^{\boldsymbol{\pi}_i}_{N_i}(\tau, \boldsymbol{x})}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} + \frac{\partial h_G(\Phi^{\boldsymbol{\pi}_i}_{N_i}(\tau_i, \boldsymbol{x}))}{\partial t} \geq -\alpha(h_G(\Phi^{\boldsymbol{\pi}_i}_{N_i}(\tau_i, \boldsymbol{x}))). \tag{21b}$$

Remark 4: In practice, encoding (21b) for all τ_i is challenging. So a simplification is adopted by only encoding $\tau_i = 0$ for the current time step, since the condition $\forall \tau_i \in [0, N_i], \Phi^{\boldsymbol{\pi}_i}_{N_i}(\tau_i, \boldsymbol{x}) \in \mathcal{C}_G$ has already been enforced as (19b) in the backup controller, and the error can be be bounded using the Lipschitz constants as discussed in Remark 3.

Remark 5: We acknowledge that our method exhibits the highest computational complexity compared to [12] and [17]. However, it is still of the same class as that of MPC-based STL planning methods [12] per iteration with respect to the dimension of control signal, which is acceptable in practice.

V. Numerical Simulation

This section presents a numerical simulation of the proposed control approach, which is performed using Python on a computer with an Intel Core i5-13600KF CPU. Consider a mobile robot model as a 2-D double integrator. The state vector is defined as $\boldsymbol{x} = [p_x, p_y, v_x, v_y]^{\top}$, where $\boldsymbol{p} = [p_x, p_y]^{\top}$ denotes the position and $\boldsymbol{v} = [v_x, v_y]^{\top}$ represents the velocity in the X and Y directions, respectively. The control input vector $\boldsymbol{u} = [a_x, a_y]^{\top}$ corresponds to the accelerations in each direction, and the desired control input is given a priori by human commander as $\boldsymbol{u}_{ref} = [2.5 + \sin \pi t, 2.5 + \cos \pi t]^{\top}$.

The simulation is conducted in a bounded 2-D environment for a duration of 6-seconds. The formula under timed-reach avoid specification is given as $\phi = \phi_G \wedge \phi_{F_1} \wedge \phi_{F_2}$ where $\phi_G := G_{[0,6]}(\| \boldsymbol{p} - [5,2]^{\top} \|^2 \geq 1.5^2)$, $\phi_{F_1} := F_{[0.2,3]}(\| \boldsymbol{p} - [2,4]^{\top} \|^2 \leq 1^2)$, $\phi_{F_2} := F_{[3.2,6]}(\| \boldsymbol{p} - [6,5]^{\top} \|^2 \leq 1^2)$.

The control method proposed in Section III is employed to synthesize a controller that satisfies ϕ, with $T = 2$s and $t_{0i} = a_i - 0.2$s. Fig. 2a shows the robot's trajectory, demonstrating that the robot successfully reaches the designated region at 2.77s and 5.76s, while always avoiding the obstacle, thus ϕ satisfied. Fig. 2b displays the backup trajectories for every 5 steps. Each backup trajectory is guaranteed to reach target regions that are considered as backup sets. Although these backup trajectories are not followed, they provide implicit safety assurance for the real-time system. Fig. 3 illustrates the desired input and the actual input. When the backup CBFs are active, the filter modifies the desired input to ensure ϕ.

VI. Conclusion

This letter incorporates timed reach-avoid specifications into safety control problems using backup control barrier

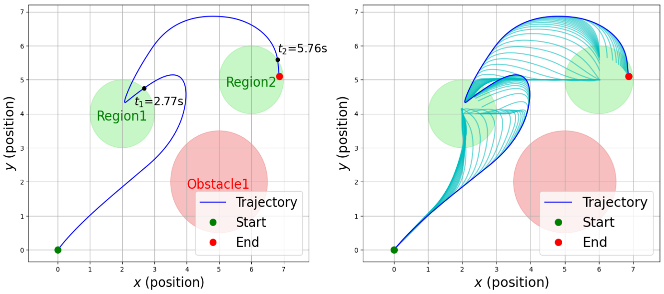


Fig. 2. (left) The trajectory generated by proposed method, where target regions are marked in green and obstacles are marked in red. (right) The backup trajectories are generated online but never be followed.

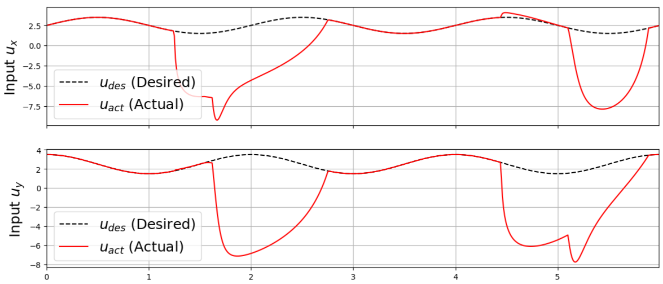


Fig. 3. Desired input (dashed line) and actual input (red line).

functions (backup CBFs). Specifically, our approach transforms the satisfaction of composed Always formulas into a global safe set, while each Eventually formula is ensured by reaching a backup set within a finite horizon. Backup CBFs enforce system trajectories remain within implicit sets that are contractive and constructed online, thus designated specifications are provably satisfied. Consequently, we offer an efficient method for encoding formal specifications into optimal controllers. This framework not only reduces the complexity associated with adjusting temporal properties within predefined performance functions, but also integrates task planning with continuous control execution. Future work will extend our framework of backup CBFs to consider control synthesis for the full class of signal temporal logic formulas.

References

[1] W. Xiao, C. G. Cassandras, and C. Belta, *Safe Autonomy With Control Barrier Functions: Theory and Applications*. Cham, Switzerland: Springer, 2023.

[2] C. Belta, B. Yordanov, and E. A. Gol, *Formal Methods for Discrete-Time Dynamical Systems*. Cham, Switzerland: Springer, 2017.

[3] M. Cohen and C. Belta, *Adaptive and Learning-Based Control of Safety-Critical Systems*. Cham, Switzerland: Springer, 2023.

[4] S. Mitra, *Verifying Cyber-Physical Systems: A Path to Safe Autonomy*. Cambridge, MA, USA: MIT Press, 2021.

[5] O. Maler and D. Nickovic et al., "Monitoring temporal properties of continuous signals," in *Proc. Int. Symp. Formal Techn. Real-Time Fault-Tolerant Syst.*, 2004, pp. 152–166.

[6] P. Akella, M. Ahmadi, R. M. Murray, and A. D. Ames, "Barrier-based test synthesis for safety-critical systems subject to timed reach-avoid specifications," *IEEE Trans. Autom. Control*, vol. 70, no. 4, pp. 2180–2195, Apr.2025.

[7] Z. Huang, W. Lan, and X. Yu, "A formal control framework of autonomous vehicle for signal temporal logic tasks and obstacle avoidance," *IEEE Trans. Intell. Vehicles*, vol. 9, no. 1, pp. 1930–1940, Jan.2024.

[8] Y. Meng and C. Fan, "Signal temporal logic neural predictive control," *IEEE Robot. Autom. Lett.*, vol. 8, no. 11, pp. 7719–7726, Nov.2023.

[9] V. Kurtz and H. Lin, "Mixed-integer programming for signal temporal logic with fewer binary variables," *IEEE Control Syst. Lett.*, vol. 6, pp. 2635–2640, 2022.

[10] G. A. Cardona, D. Kamale, and C.-I. Vasile, "Mixed integer linear programming approach for control synthesis with weighted signal temporal logic," in *Proc. 26th ACM Int. Conf. Hybrid Syst., Comput. Control*, May 2023, pp. 1–12.

[11] X. Yu, C. Wang, D. Yuan, S. Li, and X. Yin, "Model predictive control for signal temporal logic specifications with time interval decomposition," in *Proc. 62nd IEEE Conf. Decis. Control (CDC)*, Dec. 2023, pp. 7849–7855.

[12] I. Haghighi, N. Mehdipour, E. Bartocci, and C. Belta, "Control from signal temporal logic specifications with smooth cumulative quantitative semantics," in *Proc. IEEE 58th Conf. Decis. Control (CDC)*, Dec. 2019, pp. 4361–4366.

[13] R. Takano, H. Oyama, and M. Yamakita, "Continuous optimization-based task and motion planning with signal temporal logic specifications for sequential manipulation," in *Proc. IEEE Int. Conf. Robot. Autom. (ICRA)*, May 2021, pp. 8409–8415.

[14] X. Zhou, Y. Zou, S. Li, X. Li, and H. Fang, "Distributed model predictive control for multi-robot systems with conflicting signal temporal logic tasks," *IET Control Theory Appl.*, vol. 16, no. 5, pp. 554–572, Mar.2022.

[15] P. Liu, Y. He, Y. Qin, H. Zhou, and Y. Ji, "A value function space approach for hierarchical planning with signal temporal logic tasks," *IEEE Control Syst. Lett.*, vol. 9, pp. 1988–1993, 2025.

[16] H. K. Venkataraman, D. Aksaray, and P. Seiler, "Tractable reinforcement learning of signal temporal logic objectives," in *Proc. 2nd Conf. Learn. Dyn. control*, 2020, pp. 308–317.

[17] L. Lindemann and D. V. Dimarogonas, "Control barrier functions for signal temporal logic tasks," *IEEE Control Syst. Lett.*, vol. 3, no. 1, pp. 96–101, Jan.2019.

[18] A. T. Buyukkocak, D. Aksaray, and Y. Yazicioglu, "Control barrier functions with actuation constraints under signal temporal logic specifications," in *Proc. Eur. Control Conf. (ECC)*, Jul. 2022, pp. 162–168.

[19] M. Charitidou and D. V. Dimarogonas, "Receding horizon control with online barrier function design under signal temporal logic specifications," *IEEE Trans. Autom. Control*, vol. 68, no. 6, pp. 3545–3556, Jun.2023.

[20] W. Xiao and C. Belta, "High-order control barrier functions," *IEEE Trans. Autom. Control*, vol. 67, no. 7, pp. 3655–3662, Jul.2022.

[21] R. Das, A. Basu, and P. Jagtap, "Spatiotemporal tubes for temporal reach-avoid-stay tasks in unknown systems," *IEEE Trans. Autom. Control*, vol. 71, no. 1, 2026.

[22] R. Das, P. Bakshi, and P. Jagtap, "Control barrier functions for prescribed-time reach-avoid-stay tasks using spatiotemporal tubes," in *Proc. Eur. Control Conf. (ECC)*, Jun. 2025, pp. 398–403.

[23] T. Gurriet, M. Mote, A. D. Ames, and E. Feron, "An online approach to active set invariance," in *Proc. IEEE Conf. Decis. Control (CDC)*, Miami Beach, FL, USA, Dec. 2018, pp. 3592–3599.

[24] Y. Chen, M. Jankovic, M. Santillo, and A. D. Ames, "Backup control barrier functions: Formulation and comparative study," in *Proc. 60th IEEE Conf. Decis. Control (CDC)*, Dec. 2021, pp. 6835–6841.

[25] A. Singletary, P. Nilsson, T. Gurriet, and A. D. Ames, "Online active safety for robotic manipulators," in *Proc. IEEE/RSJ Int. Conf. Intell. Robots Syst. (IROS)*, Nov. 2019, pp. 173–178.

[26] D. E. J. van Wijk, S. Coogan, T. G. Molnar, M. Majji, and K. L. Hobbs, "Disturbance-robust backup control barrier functions: Safety under uncertain dynamics," *IEEE Control Syst. Lett.*, vol. 8, pp. 2817–2822, 2024.

[27] D. Ko and W. K. Chung, "A backup control barrier function approach for safety-critical control of mechanical systems under multiple constraints," *IEEE/ASME Trans. Mechatronics*, vol. 30, no. 6, pp. 4460–4471, Dec.2025.